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Fixed Point Theorem for *R*-weakly Comuting Hybrid Mappings in Metrically Convex Spaces

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ABSTRACT : In this paper we prove a fixed point theorem for two mappings of R-weakly commuting mappings in metrically convex spaces which generalizes the result due to Amit singh [1]. In process, several previous known results due to Imdad and Khan [9,10], Dolhare[5] and Nadler's [11] and others are derived as special cases.

Keywords : Fixed point, metrically convex metric spaces, hybrid contractie condition, R-weakly commuting mappings.

I. INTRODUCTION

As established in fixed point theorems for single-valued and multi-valued mappings have been studied extensively and applied to diverse problems during the last few years. Imdad and Khan [9,10], Dolhare and Petrusel [5] proved some fixed point theorems for a sequence of set valued mappings which generalize the results due to Khan [7, 8], Ahmad and Khan [3], Amit singh [1] and others. Several authors proved some fixed point theorems for self mappings. Assad and Kirk [4] gave sufficient conditions for non-self mappings to ensure the fixed point proving a result on multi-valued contractions in complete metrically convex metric spaces. The purpose of this paper is to prove some coincidence and common fixed point theorems for a sequence of hybrid type nonself mappings satisfying certain contraction condition by using *R*-weakly commutatively between multi-valued mappings and single-valued mappings.

II. PRELIMINARIES

Let (X, d) be a metric space. Then following by Nadler [11] , we recall

(*i*) $CB(X)$ = {*A*: *A* is non-empty closed and bounded subset of X}

(*ii*) $C(X)$ = {*A*: *A* is non-empty compact subset of *X*}

(*iii*) For non-empty subsets *A*, *B* of *X* and $x \in X, d(x; A) = \inf\{d(x; a) : a \in A\}$

$$H(A; B) = \max[\{\sup d(a; B) : a \in A\}; \{\sup d(A.b) : b \in B\}]$$

It is well known that $CB(X)$ is a metric space with the distance *H* which is known as Hausdro-Pompeiu metric on *X*.

The following definitions will be used in the our proof.

Definition 1.1: Let *P* be a nonempty subset of a metric space $(X, d), T : P \to X$ and $F : P \to CB(X)$. The pair (*F*, *T*) is said to be point wise *R*-weakly commuting on *P* if for given $x \in P$ and $Tx \in P$, there exists some $R = R(x) > 0$ such that $d(Ty, FTx) \leq R, d(Tx, Fx)$ for each $y \in P \cap Fx$.

Moreover, the pair (*F*, *T*) will be called R-weakly commuting on *K* if holds for each $x \in P, Tx \in P$ with some $R > 0$.

If $R = 1$, we get the definition of weak commutatively of (*F*, *T*) on *P* due to Hadzic [12, 13] and Gajic [6]. For $K = X$ reduces to "point wise *R*-weakly commutatively" for single valued self mappings .

Definition 2.2: Let *K* be a nonempty subset of a metric space $(x, d), T : K \to X$ and $F : K \to CB(X)$. The pair (*F*, *T*) is said to be quasi-coincidentally commuting if for all coincidence points "*x*" of (*T*, *F*), $TFx \in FTx$ whenever $Fx \in K$ and $Tx \in K$ for all $x \in K$.

Definition 2.3: Let (*X*, *d*) be a complete metric space and let *T* be a mapping from *X* into $CB(X)$ such that for all $x, y \in X$, $Hd(Tx, Ty) \leq rd(x, y)$ where, $0 \leq r < 1$. Then *T* has a fixed point.

Definition 2.4: Let *K* be a non-empty subset of a metric space (*X*; *d*); $T : K \to X$ and $F : K \to CB(X)$. The pair (*F*, *T*) is said to be weakly commuting if for every $x; y \in K$ with $x \in Fy$ and $Ty \in K$, we have

$d(Tx; FTy) = d(Ty; Fy)$

In this Paper, we prove the following theorem :

Amit singh [1] proved the following theorem :

Theorem *A*: Let (*X*, *d*) be a complete metrically convex metric space and *K* is nonempty closed subset of *X*.

Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S, T : K \to X$

Satisfying

(*i*) $\delta K \subseteq SK \cap TK, F_i(K) \cap K \subseteq SK, F_j(K) \cap K \subseteq TK$

(*ii*) (F_i, T) and (F_j, S) are point wise R-weakly commuting pairs.

(*iii*) $Tx \in \delta K \Rightarrow F_i(x) \subseteq K, Sx \in \delta K \Rightarrow F_j(x) \subseteq K$ and

$H[F_i(x), F_j(y)] \le ad(Tx, Sy) + b_{\max}$
$\{d(Tx, F_i(y)\}, d\{(Sy, F_j(y)\} + c_{\max}$
$\{d(Tx, Sy), d(Tx, F_i(x)\}, d\{(Sy, F_j(y)\}$

where $i = 2n - 1$, $j = 2n$, $(n \in N), i \ne j$ for all $x, y \in K$ with $x \ne y, a, b \ge 0$ and $\{(a + 2b + 2c) + (a^2 + ab + ac)/q\} < q < 1$.

(*iv*) $\{F_n\}$, S and T are continuous on K.

Then (F_i, S) and (F_j, T) have a point of coincidence.

Theorem *B*: Let (X, d) be a complete metrically convex metric space and P is nonempty closed subset of X. Let

$\{F_n\}_{n=1}^{\infty} P \to CB(X)$ and $S, T, M : P \to X$ satisfying

(*i*) $\delta P \subseteq SP \cap TP \cap MP, F_i(P) \cap P \subseteq SP,$

$F_j(P) \cap PC \subseteq TP, F_k(P) \cap P \subseteq MP$

(*ii*) (F_i, S) and (F_j, T) are point wise R-weakly commuting pairs.

(*iii*) $Tx \in \delta P \Rightarrow F_i(x) \subseteq P, Sx \in \delta P \Rightarrow F_j(x) \subseteq P$ and $H[F_i(x), F_j(y)] \le ad(Tx, Sy) + b_{\max}\{d(Tx, F_j(y)\}$

where $i = n - 1, j = n$, $(n \in I), i \ne j$ for all $x, y \in P$ with $x \ne y, a, b \ge 0$

(*iv*) $\{F_n\}$, S, T and M are continuous on P.

Then (F_i, S), (F_j, T) and (F_k, M) have a point of coincidence

Proof: Firstly we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

Let $x \in \delta P$. Since $\delta P \subseteq TP$ there exists a point $x_0 \in P$ such that $x = Sx_0$. From the implication $Sx_0 \in \delta P$ which implies $F_1(x_0) \subseteq F_1(P) \cap P \subseteq SP$.

Since $y_1 \in F_1(x_0)$ there exists a point $y_2 \in F_2(x_1)$ such that $g.d(y_1, y_2) \le H[F_1(x_0), F_2(x_1)]$.

Suppose $y_1 \in P$. Then $y_1 \in F_1(P) \cap P \subseteq SP$ implies that there exists a point $x_1 \in P$ such that $y_1 \in Sx_1$. Otherwise, if $y_1 \notin P$, then there exists a point $p \in \delta P$ such that $d(Sx_0, p) + d(p, y_1) = d(Sx_0, y_1)$.

Since $p \in \delta P \subseteq SP$, there exists a point $x_1 \in P$ with $p = Sx_1$ so that $d(Sx_0, Tx_2) + d(Tx_2, y_1) = d(Sx_0, y_1)$

Let $y_2 \in F_2(x_1)$ be such that $g.d(y_1, y_2)$, $H[F_1(x_0), F_2(x_1)]$.

Thus on repeating the foregoing arguments, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(*v*) $y_n \in F_n(x_{n-1}), y_{n+1} \in F_{n+1}(x_n)$

(*vi*) $y_n \in P \Rightarrow y_n = Tx_n$

or $y_n \notin P \Rightarrow Tx_n \in \delta P$

and $d(Sx_{n-1}, Tx_n) + d(Tx_n, y_n) = d(Sx_{n-1}, y_n)$

(*vii*) $y_{n+1} \in P \Rightarrow y_{n+1} = Sx_{n+1}$ or $y_{n+1} \notin PSx_{n+1} \in \delta P$ and $d(Tx_n, Sx_{n+1}) + d(Sx_{n+1}, y_{n+1}) = d(Tx_n, y_{n+1})$

Now we represent

$A_0 = \{Tx_i \in Tx_n\} : Tx_i = y_i$

$A_1 = \{Tx_i \in Tx_n\} : Tx_i \ne y_i$

$B_0 = \{Sx_{i+1} \in Sx_{n+1}\} : Sx_{i+1} = y_{i+1}$

$B_1 = \{Sx_{i+1} \in Sx_{n+1}\} : Tx_{i+1} \ne y_{i+1}$

First we show that $(Tx_n, Sx_{n+1}) \notin A_1 \times B_1$ and $(Sx_{n-1}, Tx_n) \notin B_1 \times A_1$.

If $Tx_n \in A_1$, then $y_2 \ne Tx_n$ and we have

$Tx_n \in \delta P$ which implies that

$y_{n+1} \in F_{n+1}(x_n) \subseteq P$. Hence $y_{n+1} = Sx_{n+1} \in B_0$

Similarly, we can say that

$(Sx_{n-1}, Tx_n) \notin B_1 \times A_1$.

Now we have the following two cases :

Case 1: If $(Tx_n, Sx_{n+1}) \in A_0 \times B_0$, then

$gd(Tx_n, Sx_{n+1}) \le H[F_{n+1}(x_n), F_n(x_{n-1})] \le ad(Tx_n, Sx_{n-1}) + b_{\max}\{d(Tx_n, F_{n+1}(x_n)), d\{(Sx_{n-1}, F_n(x_{n-1})\} \le ad(y_n, y_{n-1}) + b_{\max}\{d(y_n, y_{n+1}), d(y_{n-1}, y_n)\}$ which is also represent

$d(Tx_n, Sx_{n+1}) \le (a + b) / g.d(Sx_{n-1}, T_n)$, if $d(y_{n-1}, y_n) \ge d(y_{n+1}, y_n)$

or $d(Tx_n, Sx_{n+1}) \le hd(Sx_{n-1}, T_n)$

where $h = \max(a + b) / g < 1$

Similarly if $(S_{n-1}, Tx_n) \in B_0 \times A_0$, then

$d(S_{n-1}, Tx_n) \le (a + b) / g.d(Sx_{n-1}, T_{n-2})$, if $d(y_{n-2}, y_{n-1}) \ge d(y_{n-1}, y_n)$

or $d(Sx_{n-1}, Tx_n) \le h.d(Sx_{n-1}, T_{n-2})$

where $h = \max(a + b) / g < 1$.

Case 2: If $(Tx_n, Sx_{n+1}) \in A_0 \times B_1$, then

$d(Tx_n, Sx_{n+1}) + d(Sx_{n+1}, y_{n+1}) = d(Tx_n, y_{n+1})$

which is also represent

$d(Tx_n, Sx_{n+1}) \le d(Tx_n, y_{n+1}) = d(y_n, y_{n+1})$ and hence

$g.d(Tx_n, Sx_{n+1}) \le g.d(y_n, y_{n+1}) \le H[F_{n+1}(x_n), F_n(x_{n-1})]$.

Therefore combining above inequalities, we have

$d(Tx_n, Sx_{n+1}) \le k.d(Sx_{n-1}, Tx_{n-2})$

where $k = \max\{(a + b) / g, (g + a + b) / g\} < 1$

Similarly one can establish the other inequalities as well. Thus in all the cases we have

$d(Tx_n, Sx_{n+1}) \le k_{\max}\{d(Sx_{n-1}, Tx_n), d(Tx_{n-2}, Sx_{n-1})\}$

whereas

$d(Tx_{n+1}, Sx_{n+1}) \le k_{\max}\{d(Sx_{n-1}, Tx_n), d(Tx_n, Sx_{n-1})\}$

Now on the lines of Assad and Kirk [4], it can be shown by induction that for $n = 1$,wehave

$d(Tx_{n+1}, Sx_{n+1}) \le k^{n/2}\mu$, $d(Sx_{n+1}, Tx_{n+2}) \le k^{n/2+1}\cdot\mu$

whereas

$\mu = k^{-1}\max\{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}$

Thus the sequence

$\{Tx_0, Sx_1, Tx_2, Sx_3, \ldots Tx_n, Sx_{n+1}\}$ is a Cauchy sequence and hence converges to a point z in X. Now we assume that there exists a subsequence $\{Tx_{nk}\}$ of $\{Tx_n\}$which is contained in A_0. Further subsequences $\{Tx_{nk}\}$ and $\{Sx_{nk+1}\}$ both converge to $z \in P$ as P is closed subset of the complete metric space (X, d). Since $Tx_{nk} \in F_j(x_{nk-1})$.

For every even integers $j \in I$ and $Sx_{nk-1} \in P$ using point wise R-weakly commutatively of (F_j, S) we have

$d[SF_j(x_{nk-1}), F_j(Sx_{nk-1})] \le R_1 d[F_j(x_{nk-1}), Sx_{nk-1}]$ for every even integer $j \in I$ with some $R_1 > 0$. Also

$d[SF_j(x_{nk-1}), F_j(z)] \le d[SF_j(x_{nk-1}),$

$F_j(Sx_{nk-1})] + H[F_j(x_{nk-1}), F_j(z)]$

Making $k \to \infty$ in above two conditions and using the continuity of S and F_j, we get $d\{Sz, F_j(z)\} \le 0$ yielding thereby $Sz \in F_j(z)$, for any even integer $j \in$ I. Using point wise R-weak commutatively of (F_i, T) we have

$d\{TF_i(x_{nk})\}, F_i(Tx_{nk}) \le R_2 d(F_i(x_{nk}), Tx_{nk})$ for every odd integer $i \in I$ with some $R_2 > 0$, besides

$d[TF_i(x_{nk}), F_i(z)] \le d[TF_i(x_{nk}), F_i(\mathrm{T}x_{nk})] + H[F_i(x_{nk}), F_i(z)]$

Therefore as earlier the continuity of F_i and T implies $d\{Tz, F(z)\} \le 0$ yielding thereby $Tz \in F_i(z)$, for any odd integer $i \in I$ as $k \to \infty$.

If we assume that there exists a subsequence $\{Sx_{nk+1}\}$ contained in B_0, then above inequalities establish the earlier conclusions.

Remark

If we put $c = 0$ in theorem A then we get theorem B.

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